ON PEČARIĆ'S INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some related results to Pečarić's inequality in inner product spaces that generalises Bombieri's inequality, are given.

1. Introduction

In 1992, J.E. Pečarić [3] proved the following inequality for vectors in complex inner product spaces $(H;(\cdot,\cdot))$.

Theorem 1. *Suppose that $x, y_1, \ldots, y_n$ are vectors in H and $c_1, \ldots, c_n$ are complex numbers. Then the following inequalities*

$$\left|\sum_{i=1}^{n} c_i\left(x, y_i\right)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n}\left|c_i\right|^2\left(\sum_{j=1}^{n}\left|\left(y_i, y_j\right)\right|\right) \leq \|x\|^2 \sum_{i=1}^{n}\left|c_i\right|^2 \max_{1 \leq i \leq n}\left(\sum_{j=1}^{n}\left|\left(y_i, y_j\right)\right|\right), \tag{1.1}$$

hold.

He also showed that for $c_i=\overline{(x, y_i)}, i \in\{1, ..., n\}$, one gets

$$\left(\sum_{i=1}^{n}\left|\left(x, y_i\right)\right|^2\right)^2 \leq\|x\|^2 \sum_{i=1}^{n}\left|\left(x, y_i\right)\right|^2\left(\sum_{j=1}^{n}\left|\left(y_i, y_j\right)\right|\right) \leq\|x\|^2 \sum_{i=1}^{n}\left|\left(x, y_i\right)\right|^2 \max_{1 \leq i \leq n}\left(\sum_{j=1}^{n}\left|\left(y_i, y_j\right)\right|\right), \tag{1.2}$$

which improves Bombieri's result [1] (see also [2, p. 394])

$$\sum_{i=1}^{n}\left|\left(x, y_i\right)\right|^2 \leq\|x\|^2 \max_{1 \leq i \leq n}\left(\sum_{j=1}^{n}\left|\left(y_i, y_j\right)\right|\right). \tag{1.3}$$

Note that (1.3) is in its turn a natural generalisation of *Bessel's inequality*

$$\sum_{i=1}^{n}\left|\left(x, e_i\right)\right|^2 \leq\|x\|^2, \quad x \in H, \tag{1.4}$$

which holds for the orthornormal vectors $(e_i)_{1\leq i\leq n}$.

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In this paper we point out some related results to Pečarić's inequality (1.1). Some results of Bombieri type are also mentioned.

2. Preliminary Results

We start with the following lemma that is interesting in its own right.

Lemma 1. *Let $z_1, \dots, z_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \left(\sum_{i=1}^{n} |\alpha_i|^p \left(\sum_{j=1}^{n} |(z_i, z_j)| \right) \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\alpha_i|^q \left(\sum_{j=1}^{n} |(z_i, z_j)| \right) \right)^{\frac{1}{q}} \tag{2.1}$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum\limits_{i,j=1}^n |(z_i,z_j)|; \\[2ex] \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i,j=1}^n |(z_i,z_j)|\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \\ \hfill \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex] \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i,j=1}^n |(z_i,z_j)|\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\frac{1}{q}}; \\[2ex] \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i,j=1}^n |(z_i,z_j)|\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{\beta q}}, \\ \hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] \left(\sum\limits_{i=1}^n |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i=1}^n |\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\ \quad \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \quad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex] \left(\sum\limits_{i=1}^n |\alpha_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n |\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \\ \hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^n |\alpha_i|^{p}\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\frac{1}{p}} \left(\sum\limits_{i,j=1}^n |(z_i,z_j)|\right)^{\frac{1}{q}}; \\[2ex] \left(\sum\limits_{i=1}^n |\alpha_i|^{p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \\ \hfill \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^n |\alpha_i|^{p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |\alpha_i|^{q}\right)^{\frac{1}{q}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(z_i,z_j)|\right), \end{cases}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$.

Proof. We observe that

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 = \left(\sum_{i=1}^{n}\alpha_i z_i,\sum_{j=1}^{n}\alpha_j z_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i,z_j\right) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\left(z_i,z_j\right)\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_i\right|\left|\alpha_j\right|\left|\left(z_i,z_j\right)\right| =: M. \tag{2.2}$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^{n} m_{ij}a_{ij}b_{ij} \leq \left(\sum_{i,j=1}^{n} m_{ij}a_{ij}^{p}\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n} m_{ij}b_{ij}^{q}\right)^{\frac{1}{q}}, \tag{2.3}$$

where $m_{ij},a_{ij},b_{ij}\geq 0$, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$; then

$$M \leq \left(\sum_{i,j=1}^{n}\left|\left(z_i,z_j\right)\right|\left|\alpha_i\right|^p\right)^{\frac{1}{p}}\left(\sum_{i,j=1}^{n}\left|\left(z_i,z_j\right)\right|\left|\alpha_i\right|^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{n}\left|\alpha_i\right|^p\left(\sum_{j=1}^{n}\left|\left(z_i,z_j\right)\right|\right)\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left|\alpha_i\right|^q\left(\sum_{j=1}^{n}\left|\left(z_i,z_j\right)\right|\right)\right)^{\frac{1}{q}}, \tag{2.4}$$

and the first inequality in (2.1) is proved.

Observe that

$$\sum_{i=1}^{n}\left|\alpha_i\right|^p\left(\sum_{j=1}^{n}\left|\left(z_i,z_j\right)\right|\right) \leq \begin{cases} \max\limits_{1\leq i\leq n}\left|\alpha_i\right|^p \sum\limits_{i,j=1}^{n}\left|\left(z_i,z_j\right)\right|; \\ \left(\sum\limits_{i=1}^{n}\left|\alpha_i\right|^{\alpha p}\right)^{\frac{1}{\alpha}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}\left|\left(z_i,z_j\right)\right|\right)^{\beta}\right)^{\frac{1}{\beta}} \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \sum\limits_{i=1}^{n}\left|\alpha_i\right|^p \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}\left|\left(z_i,z_j\right)\right|\right); \end{cases}$$

giving

$$(2.5)\quad \left(\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\right)^{\frac{1}{p}}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{p}}; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\right)^{\frac{1}{\alpha p}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{\beta p}} & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^p\right)^{\frac{1}{p}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{p}}.\end{cases}$$

Similarly, we have

$$(2.6)\quad \left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\right)^{\frac{1}{q}}$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{q}} \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{\frac{1}{\gamma q}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} & \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^q\right)^{\frac{1}{q}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{q}}.\end{cases}$$

Using (2.1) and (2.5) – (2.6), we deduce the 9 inequalities in the second part of (2.2). ∎

If we choose $p=q=2$, then the following result holds.

Corollary 1. *If $z_1,\ldots,z_n\in H$ and $\alpha_1,\ldots,\alpha_n\in\mathbb{K}$, then one has*

$$(2.7)\qquad \left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2\leq\sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)$$

$$
\leq \begin{cases}
\max\limits_{1\leq i\leq n}|\alpha_i|^2 \sum\limits_{i,j=1}^{n}|(z_i,z_j)|; \\[2ex]
\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \\
\hfill \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}; \\[2ex]
\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \\
\hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}} \\
\quad\times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}} \quad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{\frac{1}{2\alpha}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{\frac{1}{2\beta}}, \\
\hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}; \\[2ex]
\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{\frac{1}{2\gamma}}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{\frac{1}{2\delta}}, \\
\hfill \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right).
\end{cases}
$$

3. Some Pečarić Type Inequalities

We are now able to point out the following result which complements and generalises the inequality (1.1) due to J. Pečarić.

Theorem 2. *Let $x, y_1, \ldots, y_n$ be vectors of an inner product space $(H; (\cdot, \cdot))$ and $c_1, \ldots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \tag{3.1}$$

$$\leq \|x\|^2 \left(\sum_{i=1}^n |c_i|^p \left(\sum_{j=1}^n |(y_i,y_j)|\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |c_i|^q \left(\sum_{j=1}^n |(y_i,y_j)|\right)\right)^{\frac{1}{q}}$$

$$\leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq i\leq n} |c_i|^2 \sum\limits_{i,j=1}^n |(y_i,y_j)|; \\[2ex] \max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i,j=1}^n |(y_i,y_j)|\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \\ \hfill \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i,j=1}^n |(y_i,y_j)|\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{q}}; \\[2ex] \max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i,j=1}^n |(y_i,y_j)|\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \\ \hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left(\sum\limits_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \left(\sum\limits_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}} \\ \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}} \quad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex] \left(\sum\limits_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \left(\sum\limits_{i=1}^n |c_i|^{\alpha p}\right)^{\frac{1}{\alpha p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{q}} \\ \quad \times \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\beta}\right)^{\frac{1}{p\beta}}, \hfill \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex] \max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{p}} \left(\sum\limits_{i,j=1}^n |(y_i,y_j)|\right)^{\frac{1}{q}}; \\[2ex] \left(\sum\limits_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |c_i|^{\gamma q}\right)^{\frac{1}{\gamma q}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right)^{\delta}\right)^{\frac{1}{\delta q}}, \\ \hfill \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \left(\sum\limits_{i=1}^n |c_i|^{p}\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^n |c_i|^{q}\right)^{\frac{1}{q}} \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^n |(y_i,y_j)|\right); \end{cases}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

Proof. We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i \right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \leq \|x\|^2 \left\| \sum_{i=1}^{n} \overline{c_i} y_i \right\|^2. \tag{3.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \ldots, n)$, we deduce the desired inequality (3.1). ∎

Remark 1. *If in (3.1) we choose* $p = q = 2$, *we obtain amongst others, the result (1.1) due to J. Pečarić.*

4. Some Results of Bombieri Type

The following results of Bombieri type hold.

Theorem 3. *Let* $x, y_1, \ldots, y_n \in H$. *Then one has the inequality:*

$$\begin{aligned}\sum_{i=1}^{n} |(x, y_i)|^2 &\\ &\leq \|x\| \left[\sum_{i=1}^{n} |(x, y_i)|^p \left(\sum_{j=1}^{n} |(y_i, y_j)| \right) \right]^{\frac{1}{2p}} \\ &\quad \times \left[\sum_{i=1}^{n} |(x, y_i)|^q \left(\sum_{j=1}^{n} |(y_i, y_j)| \right) \right]^{\frac{1}{2q}}\end{aligned} \tag{4.1}$$

$$
\leq \|x\| \times \begin{cases}
\max_{1\leq i\leq n} |(x,y_i)| \left(\sum_{i,j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2}}; \\[2ex]
\max_{1\leq i\leq n} |(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x,y_i)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \left(\sum_{i,j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\delta} \right)^{\frac{1}{2\delta q}}, \\
\hfill \text{if } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\max_{1\leq i\leq n} |(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x,y_i)|^{q} \right)^{\frac{1}{2q}} \left(\sum_{i,j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2p}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2q}}; \\[2ex]
\max_{1\leq i\leq n} |(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x,y_i)|^{\alpha p} \right)^{\frac{1}{2\alpha\beta}} \left(\sum_{i,j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2q}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\beta} \right)^{\frac{1}{p\beta}}, \\
\hfill \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\left(\sum_{i=1}^{n} |(x,y_i)|^{\alpha p} \right)^{\frac{1}{2\alpha p}} \left(\sum_{i=1}^{n} |(x,y_i)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\beta} \right)^{\frac{1}{2p\beta}} \\
\times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\delta} \right)^{\frac{1}{2\delta q}} \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\left(\sum_{i=1}^{n} |(x,y_i)|^{q} \right)^{\frac{1}{2q}} \left(\sum_{i=1}^{n} |(x,y_i)|^{\alpha p} \right)^{\frac{1}{2\alpha p}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2p}} \\
\qquad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\beta} \right)^{\frac{1}{2p\beta}}, \hfill \text{if } \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[2ex]
\max_{1\leq i\leq n} |(x,y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x,y_i)|^{p} \right)^{\frac{1}{2p}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2p}} \left(\sum_{i,j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2q}}; \\[2ex]
\left(\sum_{i=1}^{n} |(x,y_i)|^{p} \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} |(x,y_i)|^{\gamma q} \right)^{\frac{1}{2\gamma q}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2p}} \\
\qquad \times \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\delta} \right)^{\frac{1}{2\delta q}}, \hfill \text{if } \gamma > 1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\[2ex]
\left(\sum_{i=1}^{n} |(x,y_i)|^{p} \right)^{\frac{1}{2p}} \left(\sum_{i=1}^{n} |(x,y_i)|^{q} \right)^{\frac{1}{2q}} \max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |(y_i,y_j)| \right)^{\frac{1}{2}},
\end{cases}
$$

where $p > 1, \frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof follows by Theorem 2 on choosing $c_i = \overline{(x, y_i)}, i \in \{1, ..., n\}$ and taking the square root in both sides of the inequalities involved. We omit the details. ∎

Remark 2. *We observe, by the last inequality in (4.1), we get*

$$\frac{\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2}{\left(\sum_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |(x, y_i)|^q\right)^{\frac{1}{q}}} \le \|x\|^2 \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right),$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

If in this inequality we choose $p = q = 2$, *then we recapture Bombieri's result* (1.3).

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